Exercise 6

A triangle has vertices $(0, 0, 0)$, $(1, 1, 1)$, and $(0, -2, 3)$. Find its area.

Solution

Let $\mathbf{r}_1$ and $\mathbf{r}_2$ be the displacement vectors from $(0, 0, 0)$ to $(1, 1, 1)$ and $(0, -2, 3)$, respectively.

$$\mathbf{r}_1 = (1, 1, 1) - (0, 0, 0) = (1, 1, 1)$$
$$\mathbf{r}_2 = (0, -2, 3) - (0, 0, 0) = (0, -2, 3)$$

Calculate the cross product of $\mathbf{r}_1$ and $\mathbf{r}_2$.

$$
\begin{aligned}
\mathbf{r}_1 \times \mathbf{r}_2 &= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ 1 & 1 & 1 \\ 0 & -2 & 3 \end{vmatrix} \\
&= \begin{vmatrix} 1 & 1 \\ -2 & 3 \end{vmatrix} \hat{\mathbf{x}} - \begin{vmatrix} 1 & 1 \\ 0 & 3 \end{vmatrix} \hat{\mathbf{y}} + \begin{vmatrix} 1 & 1 \\ 0 & -2 \end{vmatrix} \hat{\mathbf{z}} \\
&= (3 + 2)\hat{\mathbf{x}} - (3 - 0)\hat{\mathbf{y}} + (-2 - 0)\hat{\mathbf{z}} \\
&= 5\hat{\mathbf{x}} - 3\hat{\mathbf{y}} - 2\hat{\mathbf{z}} \\
&= (5, -3, -2)
\end{aligned}
$$

The area of the parallelogram with sides $\mathbf{r}_1$ and $\mathbf{r}_2$ is given by the magnitude of this cross product.

$$A = \|\mathbf{r}_1 \times \mathbf{r}_2\| = \sqrt{5^2 + (-3)^2 + (-2)^2} = \sqrt{38}$$

The area of the triangle is half of this.

$$A_T = \frac{\sqrt{38}}{2}$$